

A_∞ -ALGEBRAS ASSOCIATED WITH ELLIPTIC CURVES AND EISENSTEIN-KRONECKER SERIES

ALEXANDER POLISHCHUK

ABSTRACT. We compute the A_∞ -structure on the self-Ext algebra of the vector bundle G over an elliptic curve of the form $G = \bigoplus_{i=1}^r P_i \oplus \bigoplus_{j=1}^s L_j$, where (P_i) and (L_j) are line bundles of degrees 0 and 1, respectively. The answer is given in terms of Eisenstein-Kronecker numbers $(e_{a,b}^*(z, w))$. The A_∞ -constraints lead to quadratic polynomial identities between these numbers, allowing to express them in terms of few ones. Another byproduct of the calculation is the new representation for $e_{a,b}^*(z, w)$ by rapidly converging series.

INTRODUCTION

Let $\Lambda \subset \mathbb{C}$ be a lattice. Let $a(\Lambda)$ denote the area of the fundamental parallelogram of Λ , and set $A(\Lambda) = \frac{a(\Lambda)}{\pi}$. Recall (see [15, ch. VIII]) that the *Eisenstein-Kronecker-Lerch series* are given by

$$K_a^*(z, w, s; \Lambda) = \sum_{\lambda \in \Lambda \setminus \{-z\}} \frac{(\bar{z} + \bar{\lambda})^a}{|z + \lambda|^{2s}} \langle \lambda, w \rangle_\Lambda,$$

where $a \in \mathbb{Z}_{\geq 0}$, $z, w \in \mathbb{C} \setminus \Lambda$, s is a real number,

$$\langle z, w \rangle_\Lambda := \exp[(z\bar{w} - w\bar{z})/A(\Lambda)].$$

This series converges absolutely for $\operatorname{Re} s > a/2 + 1$. It is known that $K_a^*(z, w, s; \Lambda)$ analytically extends (for fixed z, w) to a meromorphic function on the entire s -plane, with possible poles only at $s = 0$ (for $a = 0$, $z \in \Lambda$) and at $s = 1$ (for $a = 0$, $w \in \Lambda$). Using this analytical continuation the *Eisenstein-Kronecker numbers* $e_{a,b}^*(z, w; \Lambda)$, for integers $a \geq 0$, $b > 0$, are defined as the following special values:

$$e_{a,b}^*(z, w) = K_{a+b}^*(z, w, b; \Lambda).$$

Note that $e_{a,b}^*(z, w)$ descends to a section of a line bundle on $C \times C$, where $C = \mathbb{C}/\Lambda$ is the corresponding elliptic curve. This section is not a continuous, however, it has continuous (even real analytic) restrictions to all the strata of the stratification of $C \times C$ induced by the stratification $C = \{0\} \sqcup (C \setminus \{0\})$.

We refer to [15, ch. VIII] and [1] for basic properties of Eisenstein-Kronecker numbers. Note that they are of number-theoretic interest because of their close connection to special values of Hecke's L-functions associated with quadratic imaginary fields (for this one takes lattices Λ with complex multiplication).

In this paper, generalizing the works [9] and [11], we show that the Eisenstein-Kronecker numbers appear as the structure constants of some natural A_∞ -algebra associated with an

Supported in part by the NSF grant DMS-1400390.

elliptic curve. As a consequence we derive quadratic relations between them which allow to express all of them polynomially in terms of the few simple ones, which are related to the usual Eisenstein series, the Weierstrass functions ζ , \wp and \wp' , and the Kronecker functions $F(z, w)$ and $\partial_z F(z, w)$ (see Corollary B below). We also get a new representation for Eisenstein-Kronecker numbers by certain rapidly converging series that appear naturally when computing the relevant A_∞ -products (see Theorem C).

We refer to [4] for an introduction to A_∞ -algebras (in particular, we use the same sign convention). For purposes of our paper the main point is the role of an A_∞ -structure as a “regularized” version of Massey products on cohomology of dg-algebras (see Sec. 2.1). In particular, any Ext-algebra $\text{Ext}^*(G, G)$ of a coherent sheaf G on an algebraic variety carries such an A_∞ -structure. There is a choice involved in constructing it, but the result is unique up to higher homotopies.

In this paper we consider two collections of (pairwise non-isomorphic) line bundles on the complex elliptic curve $C = \mathbb{C}/\Lambda$: $(P_i)_{i=1, \dots, r}$ and $(L_k)_{j=1, \dots, s}$, where $\deg P_i = 0$ and $\deg L_j = 1$. We set

$$G = \bigoplus_{i=1}^r P_i \oplus \bigoplus_{j=1}^s L_j$$

and consider the A_∞ -algebra structure on $\text{Ext}^*(G, G)$ constructed using the natural hermitian metrics on the relevant line bundles (see Sec. 2.1 for details).

Assume that $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$, where τ is in the upper half-plane. We denote by L the holomorphic line bundle of degree 1 on C , such that the classical theta-function

$$\theta = \theta(z, \tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i \tau n^2 + 2\pi i n z)$$

descends to a global section of L . The line bundles (P_i) , (L_j) can be written in the form

$$P_i = t_{w_i}^* L \otimes L^{-1}, \quad L_j = t_{z_j}^* L,$$

for some complex numbers (w_i) , (z_j) (unique modulo Λ).

We fix a generator $\xi \in H^1(C, \mathcal{O})$ which is represented by $d\bar{z}$ in the Dolbeault complex. We denote by $\eta \in H^1(C, L^{-1})$ the unique generator such that $\eta \circ \theta = \xi$.

The space $\text{Ext}^*(G, G)$ has the following natural basis:

- (i) identity elements in $\text{Hom}(P_i, P_i)$, $\text{Hom}(L_j, L_j)$;
- (ii) the elements $\xi_i \in \text{Ext}^1(P_i, P_i)$ and $\xi_j \in \text{Ext}^1(L_j, L_j)$, corresponding to ξ under the natural isomorphism with $H^1(C, \mathcal{O})$;
- (iii) $\theta_{ij} := t_{z_j - w_i}^* \theta \in H^0(t_{z_j - w_i}^* L) \simeq \text{Hom}(P_i, L_j)$;
- (iv) $\eta_{ji} := t_{z_j - w_i}^* \eta \in H^1(t_{z_j - w_i}^* L^{-1}) \simeq \text{Ext}^1(L_j, P_i)$.

Let us set $A = A(\Lambda)$. It is convenient to consider the following rescaling of the elements (ii)–(iv) of the above basis:

$$(0.0.1) \quad \begin{aligned} \tilde{\xi}_i &= A^{-1} \xi_i, \quad \tilde{\xi}_j = A^{-1} \xi_j, \quad \tilde{\theta}_{ij} = \exp(A^{-1} \frac{(z_j - w_i + \bar{w}_i)^2}{2}) \theta_{ij}, \\ \tilde{\eta}_{ji} &= A^{-1} \exp(-A^{-1} \frac{(z_j - w_i + \bar{w}_i)^2}{2}) \eta_{ji}. \end{aligned}$$

Note also that our A_∞ -structure on $\text{Ext}^*(G, G)$ is *cyclic* with respect to a natural pairing $\langle \cdot, \cdot \rangle$ on $\text{Ext}^*(G, G)$ (see Sec. 2.1, Eq. (2.1.1)).

Now we can describe our answer for the A_∞ -structure on $\text{Ext}^*(G, G)$.

Theorem A. *For $a, b, c, d \geq 0$ one has*

$$m_n((\tilde{\xi}_i)^a, \tilde{\theta}_{ij}, (\tilde{\xi}_j)^b, \tilde{\eta}_{ji'}, (\tilde{\xi}_{i'})^c, \tilde{\theta}_{i'j'}, (\tilde{\xi}_{j'})^d) = (-1)^{\binom{n+1}{2}+1} \frac{(b+d)!}{a!b!c!d!A^{a+c}} \cdot e_{a+c, b+d+1}^*(z_{j'} - z_j, w_i - w_{i'}) \cdot \tilde{\theta}_{ij'}.$$

Note that here the indices in the pairs (i, i') and (j, j') are not necessarily distinct. The remaining m_n are determined by the condition that our A_∞ -structure on $\text{Ext}^*(G, G)$ is cyclic (see Lemma 2.1.1).

The case $r = 1$ in Theorem A corresponds to the A_∞ -algebra associated with an elliptic curve equipped with n distinct points p_1, \dots, p_n considered in [6]. Indeed, the corresponding bundle $G = \mathcal{O}_C \oplus \bigoplus_{j=1}^s L_j$ is related to the object $\mathcal{O}_C \oplus \bigoplus_{j=1}^s \mathcal{O}_{p_j}$ considered in [6] by an autoequivalence of $D^b(\text{Coh } C)$.

Theorem A implies that the numbers $e_{a,b}^*(z, w)$ satisfy many quadratic equations (see Proposition 2.3.1). This leads in particular to the following result.

Corollary B. *For any $a, b \geq 0$ there exist polynomials $P_{a,b}$ with rational coefficients, such that for any $z, w \notin \Lambda$ one has*

$$\frac{e_{a,b}^*(z, w)}{A^a} = P_{a,b}(e_{0,1}^*(z, w), e_{0,2}^*(z, w), E_1^*(z), E_2^*(z), E_3^*(z), E_1^*(w), E_2^*(w), E_3^*(w), e_2^*, e_4, e_6),$$

where $E_n^*(z) = e_{0,n}^*(z, 0)$.

Note that here $e_{0,1}^*(z, w)$ differs by a simple exponential factor from the Kronecker function $F(z, w)$ (see (1.1.6), (1.1.7)), $e_{0,2}^*(z, w) = -\partial_z e_{0,1}^*(z, w)$, $E_1^*(z) = Z(z)$ is the modified Weierstrass zeta-function (see (1.1.9)), $E_2^*(z)$ differs from $\wp(z)$ by a constant, and $E_3^*(z) = -\wp'(z)/2$. The polynomials $P_{a,b}$ can be computed by an explicit recursion (see Corollary 2.3.2).

For $w = 0$ (resp., $w = z = 0$) the result of Corollary B, that $e_{a,b}^*(z, 0)/A^a$ (resp., $e_{a,b}^*(0, 0)/A^a$) can be expressed as a polynomial with rational coefficients in $E_1^*(z)$, $E_2^*(z)$, $E_3^*(z)$, e_2^* , e_4 and e_6 , follows from [2, Prop. 9] (resp., [15, VI.5, p. 45]).

Now let us describe our new series for the Eisenstein-Kronecker numbers. For $m, n \in \mathbb{Z}$, a lattice $\Lambda \subset \mathbb{C}$ and $z, w \in C = \mathbb{C}/\Lambda$ we set

$$(0.0.2) \quad f_{m,n}^*(z, w) = f_{m,n}^*(z, w, \Lambda) = A^{-m} \cdot \sum_{\lambda \in \Lambda \setminus \{-z\}} \frac{(\bar{\lambda} + \bar{z})^m}{(\lambda + z)^n} \exp(-A^{-1}|\lambda + z|^2) \langle w, \lambda \rangle.$$

As in the case of $e_{a,b}^*(z, w)$, we use the notation f^* to stress the fact that one should really consider restrictions of this function to the strata of $C \times C$ induced by $C = \{0\} \sqcup (C \setminus \{0\})$.

Theorem C. *One has*

$$e_{a,b+1}^*(z, -w) = \frac{A^a}{b!} \cdot \sum_{k \geq 0} k! \left((-1)^{a+b+1} \binom{a}{k} f_{a+b-k, k+1}^*(w, z) \cdot \langle z, w \rangle_\Lambda + \binom{b}{k} f_{a+b-k, k+1}^*(z, w) \right).$$

Note that in the case when all z_j and w_i are zero (resp., $z = w = 0$) Theorems A and C were proved in [11]. Also, the case of m_3 in Theorem A (corresponding to $a = b = c = d = 0$) and the identity of Theorem C for $a = b = 0$ were established in [9]. The method of computation

of the general A_∞ -products in this paper is similar to those in [9] and [11]. However, the answer appears in the form of a rapidly converging series as in the right-hand side of the formula of Theorem C. It took the author a while to realize that these series are related to the Eisenstein-Kronecker series.

Using Theorem C we get a simpler derivation of the fact that a certain version of the Kronecker function F gives a generating series for the Eisenstein-Kronecker numbers, proved in [1, Thm. 1.17] (see Proposition 1.2.2).

It is interesting to study how the A_∞ -structure on $\text{Ext}^*(G, G)$ computed in Theorem A varies when we vary the parameters (w_i) and (z_j) determining the line bundles P_i and L_j (and keeping the lattice Λ fixed). There are two phenomena that we can observe using our explicit formulas. First, the dependence of the gauge equivalence class of the A_∞ -structure on (w_i) and (z_j) should be holomorphic. The reason for this is that there is an alternative, purely algebraic way to construct the same gauge equivalence class of A_∞ -structure using Čech resolutions (see e.g., [12, Sec. 3] for a similar construction). This means that the derivatives $\partial_{\overline{w}_i}$ and $\partial_{\overline{z}_j}$ of the products (m_\bullet) , after an appropriate choice of a basis in $\text{Ext}^*(G, G)$, define the trivial class in the truncated Hochschild cohomology $HH_{\leq 0}^2$ of the A_∞ -algebra given by (m_\bullet) . In other words we should have equations of the form

$$\partial_{\overline{w}_i}(m_\bullet) = [b_i, (m_\bullet)], \quad \partial_{\overline{z}_j}(m_\bullet) = [b_j, (m_\bullet)],$$

for appropriate 1-cochains b_i and b_j in the truncated Hochschild complex $CH_{\leq 0}^*$, where $[\cdot, \cdot]$ denotes the Gerstenhaber bracket. We show that in our case there is a very simple choice of (b_i) and (b_j) (see Proposition 2.4.1).

Secondly, the fact that all the A_∞ -algebras in our family are derived Morita equivalent also has an infinitesimal incarnation. Namely, the infinitesimal variation of the derived Morita equivalence class is captured by the full Hochschild cohomology HH^2 of the A_∞ -algebra. Hence, for our family we should have equations of the form

$$\partial_{w_i}(m_\bullet) = [c_i, (m_\bullet)], \quad \partial_{z_j}(m_\bullet) = [c_j, (m_\bullet)],$$

for appropriate 1-cochains c_i and c_j in the full Hochschild complex CH^* . We show (see Proposition 2.4.1) that this is indeed the case for a very simple choice of 1-cochains. Note that similar kind of structures appear in symplectic geometry under the name of *pseudo-isotopies*, when one considers the variation of the Fukaya products with respect to a variation of an almost complex structure (see [3], [14]).

1. NEW SERIES FOR THE EISENSTEIN-KRONECKER NUMBERS

1.1. Basic properties of the Eisenstein-Kronecker numbers. First, let us recall some properties of $e_{a,b}^*(z, w)$. We fix a lattice Λ and set $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_\Lambda$, $A = A(\Lambda)$.

Directly from the definition (using analytic continuation) we get

$$e_{a,b}^*(-z, -w) = (-1)^{a+b} e_{a,b}^*(z, w),$$

$$(1.1.1) \quad e_{a,b}^*(z, w + \lambda) = e_{a,b}^*(z, w), \quad e_{a,b}^*(z + \lambda, w) = e_{a,b}^*(z, w) \cdot \langle \lambda, w \rangle^{-1}, \quad \text{for } \lambda \in \Lambda,$$

$$(1.1.2) \quad e_{a,b}^*(0, w) = \lim_{z \rightarrow 0} \left[e_{a,b}^*(z, w) - \frac{\bar{z}^a}{z^b} \right]$$

(cf. [15, VIII.15]).

The functional equation for the Eisenstein-Kronecker-Lerch series (see [15, VIII.13, Eq. (32)]) implies the following identity

$$(1.1.3) \quad (b-1)!e_{a,b}^*(z, w) = A^{a-b+1} \cdot a!e_{b-1,a+1}^*(w, z) \cdot \langle w, z \rangle,$$

for $a \geq 0, b \geq 1$. We also have

$$(1.1.4) \quad \partial_z e_{a,b}^*(z, w) = -b e_{a,b+1}^*(z, w), \quad \partial_w e_{a,b}^*(z, w) = -A^{-1} e_{a+1,b}^*(z, w) + A^{-1} \bar{z} e_{a,b}^*(z, w),$$

where in the first formula $z \notin \Lambda$ and in the second formula $w \notin \Lambda$. It is also known that $e_{0,b}^*(0, 0) = 0$ for odd b , and $e_{0,2k}^*(0, 0) = e_{2k}^*$, for $k \geq 1$, where for $k \geq 2$, $e_{2k}^* = e_{2k}$ is the Eisenstein series

$$e_{2k}(\lambda_1, \lambda_2) = \sum_{\lambda \in L \setminus \{0\}} \frac{1}{\lambda^{2k}}$$

while $e_{0,2}^*(0, 0) = e_2^*$ is the modified Eisenstein series

$$e_2^* = \sum_{m \in \mathbb{Z}} \sum_{n \neq 0 \text{ if } m=0} \frac{1}{(m\lambda_2 + n\lambda_1)^2} - A^{-1} \frac{\bar{\lambda}_1}{\lambda_1},$$

where $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$. More generally, for $b > a \geq 0$ one has

$$(1.1.5) \quad e_{a,b}^*(0, 0) = \frac{(b-a-1)!}{(b-1)!} (-\mathcal{D})^a (e_{b-a}^*),$$

where $\mathcal{D} = \bar{\lambda}_1 \partial_{\lambda_1} + \bar{\lambda}_2 \partial_{\lambda_2}$ (see [15, VIII.15]).

Some of the Eisenstein-Kronecker numbers are related to other classical functions. For example, by [15, VIII.2, Eq.(3), p. 70], for $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ and $z, w \in \mathbb{C} \setminus \Lambda$, we have

$$(1.1.6) \quad e_{0,1}^*(z, w) = 2\pi i \exp(A^{-1}z(w - \bar{w})) \cdot F(z, w, \tau),$$

where $F(z, w, \tau)$ is the *Kronecker function*:

$$(1.1.7) \quad F(z, w, \tau) = - \sum_{m \geq 0, n \geq 0} \exp(2\pi i(mn\tau + mz + nw)) + \sum_{m < 0, n < 0} \exp(2\pi i(mn\tau + mz + nw)).$$

Also, we have (see [15, VIII.14]) for $z \notin \Lambda$

$$(1.1.8) \quad e_{0,1}^*(z, 0) = Z(z, \Lambda),$$

where $Z(z, \Lambda)$ is the Λ -periodic modification of the Weierstrass zeta-function:

$$(1.1.9) \quad \begin{aligned} Z(z, \Lambda) &:= \zeta(z; \lambda_1, \lambda_2) - 2z_1 \zeta\left(\frac{\lambda_1}{2}\right) - 2z_2 \zeta\left(\frac{\lambda_2}{2}\right), \\ \zeta(z) &:= \zeta(z, \lambda_1, \lambda_2) = \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{z+\lambda} - \frac{1}{\lambda} + \frac{z}{\lambda^2} \right), \end{aligned}$$

where $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$, $z = z_1\lambda_1 + z_2\lambda_2$ with $z_1, z_2 \in \mathbb{R}$.

1.2. Rapidly converging series representation. It will be convenient to consider the following modified versions of the functions $f_{m,n}^*(z, w)$ (see (0.0.2)):

$$\tilde{f}_{m,n}^*(z, w) = A^{-m} \cdot \exp(A^{-1}z(w - \bar{w})) \sum_{\lambda \in \Lambda \setminus \{-z\}} \frac{(\bar{\lambda} + \bar{z})^m}{(\lambda + z)^n} \exp(-A^{-1}|\lambda + z|^2) \langle w, \lambda \rangle.$$

Set

$$(1.2.1) \quad \begin{aligned} \tilde{g}_{a,b}^*(z, w) &= \sum_{k \geq 0} k! \left(\binom{a}{k} \tilde{f}_{a+b-k, k+1}^*(w, z) + (-1)^{a+b+1} \binom{b}{k} \tilde{f}_{a+b-k, k+1}^*(z, w) \right), \\ g_{a,b}^*(z, w) &= \exp(-A^{-1}z(w - \bar{w})) \tilde{g}_{a,b}^*(z, w) = \\ &= \sum_{k \geq 0} k! \left(\binom{a}{k} f_{a+b-k, k+1}^*(w, z) \cdot \langle z, w \rangle + (-1)^{a+b+1} \binom{b}{k} f_{a+b-k, k+1}^*(z, w) \right). \end{aligned}$$

Immediately from the definition we get the identities

$$(1.2.2) \quad g_{a,b}^*(z, w + \lambda) = g_{a,b}^*(z, w), \quad g_{a,b}^*(z + \lambda, w) = g_{a,b}^*(z, w) \cdot \langle \lambda, w \rangle,$$

$$(1.2.3) \quad \begin{aligned} \tilde{g}_{a,b}^*(z, w) &= (-1)^{a+b+1} \tilde{g}_{b,a}^*(w, z), \\ g_{a,b}^*(z, w) &= (-1)^{a+b+1} g_{b,a}^*(w, z) \cdot \langle z, w \rangle. \end{aligned}$$

Note also that

$$f_{m,n}^*(-z, -w) = (-1)^{m+n} f_{m,n}^*(z, w), \quad g_{m,n}^*(-z, -w) = (-1)^{m+n+1} g_{m,n}^*(z, w).$$

The following two identities, that are reformulations of [9, Thm. 1, Thm. 2], will play a key role for us.

$$(1.2.4) \quad g_{0,0}^*(z, 0) = \tilde{g}_{0,0}^*(z, 0) = f_{0,1}^*(0, z) - f_{0,1}^*(z, 0) = -Z(z, \Lambda),$$

$$(1.2.5) \quad \tilde{g}_{0,0}^*(z, w) = -2\pi i F(z, -w, \Lambda),$$

where $z, w \in \mathbb{C} \setminus \Lambda$.

Lemma 1.2.1. *In all of the formulas below, when computing ∂_z or $\partial_{\bar{z}}$ (resp., ∂_w or $\partial_{\bar{w}}$) we assume that $z \notin \Lambda$ (resp., $w \notin \Lambda$).*

(i) One has

$$(1.2.6) \quad \begin{aligned} \partial_z f_{m,n}^*(z, w) &= -f_{m+1,n}^*(z, w) - n f_{m,n+1}^*(z, w), \\ \partial_w f_{m,n}^*(z, w) &= f_{m+1,n}^*(z, w) - A^{-1} \bar{z} \cdot f_{m,n}^*(z, w); \\ A \partial_{\bar{z}} f_{m,n}^*(z, w) &= m f_{m-1,n}^*(z, w) - f_{m,n-1}^*(z, w), \\ A \partial_{\bar{w}} f_{m,n}^*(z, w) &= -f_{m,n-1}^*(z, w) + A^{-1} z \cdot f_{m,n}^*(z, w), \end{aligned}$$

where in the last two formulas $n \geq 1$.

(ii) One has

$$(1.2.7) \quad \partial_z g_{a,b}^*(z, w) = g_{a,b+1}^*(z, w),$$

(iii) Assume that either $w \notin \Lambda$, $b \geq 0$; or $w \in \Lambda$, $b \geq 1$. Then the function $g_{0,b}^*(z, w)$ is holomorphic in z varying in $\mathbb{C} \setminus \Lambda$. For $z \notin \Lambda$ one has

$$A \partial_{\bar{z}} g_{0,0}^*(z, w) = \delta_\Lambda(w),$$

where δ_Λ is the delta-function of the lattice Λ .

(iv) For $a \geq 0, b \geq 0$ one has

$$(1.2.8) \quad A\partial_{\bar{z}}g_{a,b}^*(z, w) = -ag_{a-1,b}^*(z, w) + \delta_{a+b,0}\delta_{\Lambda}(w).$$

(v) One has

$$(1.2.9) \quad \begin{aligned} \partial_w g_{a,b}^*(z, w) &= -g_{a+1,b}^*(z, w) - A^{-1}\bar{z}g_{a,b}^*(z, w), \\ A\partial_{\bar{w}}g_{a,b}^*(z, w) &= bg_{a,b-1}^*(z, w) + zg_{a,b}^*(z, w) - \delta_{a+b,0}\delta_{\Lambda}(z)\langle z, w \rangle. \end{aligned}$$

(vi) Let $D \subset \mathbb{C}$ be a small disk around 0 such that $D \cap \Lambda = \{0\}$. For $a, b \geq 0$, the function $g_{a,b}^*(z, w) + (-1)^{a+b}A^{-a}b! \cdot \frac{\bar{z}^a}{z^{b+1}}$ extends to a real analytic function on $D \times (\mathbb{C} \setminus \Lambda)$ and

$$(1.2.10) \quad \left[g_{a,b}^*(z, w) + (-1)^{a+b}A^{-a}b! \cdot \frac{\bar{z}^a}{z^{b+1}} \right]_{z=0} = g_{a,b}^*(0, w).$$

Proof. (i) This is a straightforward computation. Note that one has to consider separately the case when $w \in \Lambda$ (for computing ∂_z and $\partial_{\bar{z}}$) and the case $z \in \Lambda$ (for computing ∂_w and $\partial_{\bar{w}}$).

(ii) Using the definition and part (i) we get

$$\begin{aligned} \partial_z g_{a,b}^*(z, w) &= \sum_{k \geq 0} \left(k! \binom{a}{k} f_{a+b-k+1,k+1}^*(w, z) \cdot \langle z, w \rangle \right. \\ &\quad \left. - (-1)^{a+b+1} k! \binom{b}{k} f_{a+b-k+1,k+1}^*(z, w) - (-1)^{a+b+1} (k+1)! \binom{b}{k} f_{a+b-k,k+2}^*(z, w) \right). \end{aligned}$$

Changing the summation variable in the last term to $k+1$ and using the identity $\binom{b+1}{k} = \binom{b}{k} + \binom{b}{k-1}$, we immediately identify the result with $g_{a,b+1}^*$.

(iii) Assume first that $w \notin \Lambda$. Then the formula (1.2.5) shows that $\tilde{g}_{0,0}^*(z, w)$ is holomorphic in z , hence, the same is true for $g_{0,0}^*(z, w)$. By part (ii), $g_{0,b}^*(z, w)$ is obtained by applying ∂_z to $g_{0,0}^*(z, w)$ b times, hence, $g_{0,b}^*(z, w)$ is also holomorphic.

Now assume that $w = 0$ (the general $w \in \Lambda$ is reduced to $w = 0$). The formula (1.2.4) identifies $-g_{0,0}^*(z, 0)$ with $Z(z)$, which gives the required formula for $\partial_{\bar{z}}$ (using the Legendre period relation). Since $Z(z)$ differs from the usual Weierstrass zeta-function $\zeta(z)$ by linear terms, we see that $g_{0,1}^*(z, 0) = \partial_z g_{0,0}^*(z, 0)$ is already holomorphic in z (and differs from the Weierstrass \wp -function by a constant). Hence, $g_{0,b}^*(z, 0) = \partial_z^{b-1} g_{0,1}^*(z, 0)$ is also holomorphic.

(iv) For $a = b = 0$ the assertion follows from (iii), so we can assume that $a + b \geq 1$. We have

$$\begin{aligned} -A\partial_{\bar{z}}g_{a,b}^*(z, w) &= \sum_{k \geq 0} \left(k! \binom{a}{k} f_{a+b-k,k}^*(w, z) \cdot \langle z, w \rangle \right. \\ &\quad \left. + (-1)^{a+b+1} k! \binom{b}{k} f_{a+b-k,k}^* - (-1)^{a+b+1} k! (a+b-k) \binom{b}{k} f_{a+b-k-1,k+1}^*(z, w) \right). \end{aligned}$$

Changing the summation variable in the last term to $k+1$ and using the identity

$$k! \binom{b}{k} - (k-1)! (a+b-k+1) \binom{b}{k-1} = -(k-1)! a \binom{b}{k-1},$$

we can rewrite this as

$$(1.2.11) \quad -A\partial_{\bar{z}}g_{a,b}^*(z, w) = ag_{a-1,b}^* + f_{a+b,0}^*(w, z) \cdot \langle z, w \rangle + (-1)^{a+b+1} f_{a+b,0}^*(z, w).$$

Note that this still works for $a = 0$ and shows that

$$-A\partial_{\bar{z}}g_{0,b}^*(z, w) = f_{b,0}^*(w, z) \cdot \langle z, w \rangle + (-1)^{b+1}f_{b,0}^*(z, w).$$

By part (iii), this is zero for $b \geq 1$ and for $b = 0$, $w \notin \Lambda$. Hence, for $a + b \geq 1$ we can simplify the right-hand side of (1.2.11) to $ag_{a-1,b}^*$.

(v) This follows from (ii) and (iv) using (1.2.3).

(vi) First, we observe that by definition of $g_{a,b}^*(z, w)$ there exists a polynomial expression $P_{a,b}(\bar{z}, |z|^2)$ such that $g_{a,b}^*(z, w) + P_{a,b}(\bar{z}, |z|^2)z^{-b-1}$ extends to a real analytic function on $D \times (\mathbb{C} \setminus \Lambda)$, and

$$(1.2.12) \quad \left[g_{a,b}^*(z, w) + P_{a,b}(\bar{z}, |z|^2)z^{-b-1} \right] |_{z=0} = g_{a,b}^*(0, w).$$

Furthermore, in the case $a = b = 0$ we just have $P_{0,0} = 1$, i.e., $g_{0,0}^*(z, w) + z^{-1}$ is real analytic on $D \times (\mathbb{C} \setminus \Lambda)$. Note that by (ii) and (v), we have

$$(1.2.13) \quad g_{a,b}^*(z, w) = \partial_z^b(-\partial_w - A^{-1}\bar{z})^a g_{0,0}^*(z, w).$$

It follows that $g_{a,b}^*(z, w) + (-1)^{a+b}A^{-a}b! \cdot \frac{\bar{z}^a}{z^{b+1}}$ is real analytic on $D \times (\mathbb{C} \setminus \Lambda)$. This implies that the expression

$$\frac{(-1)^{a+b}A^{-a}b!\bar{z}^a - P_{a,b}(\bar{z}, |z|^2)}{z^{b+1}}$$

extends to a real analytic function on $D \times (\mathbb{C} \setminus \Lambda)$. Since the numerator is a polynomial in \bar{z} and $z\bar{z}$, this implies that

$$\left[\frac{(-1)^{a+b}A^{-a}b!\bar{z}^a - P_{a,b}(\bar{z}, |z|^2)}{z^{b+1}} \right] |_{z=0} = 0.$$

Together with (1.2.12) this implies (1.2.10). \square

Now we are ready to prove Theorem C which can be rewritten as

$$(1.2.14) \quad (-1)^{a+b+1} \frac{A^a}{b!} g_{a,b}^*(z, w) = e_{a,b+1}^*(z, -w).$$

Proof of Theorem C. First, we note that (1.2.14) holds for $z = w = 0$. Indeed, for $b \geq a \geq 0$ this follows from [11, Cor. 1.2.2], using (1.1.5). The case $b < a$ follows using the functional equation (1.1.3) together with (1.2.3).

Note also that the equation (1.2.14) is invariant with respect to the shifts of z and w by elements of Λ , due to (1.1.1) and (1.2.2).

Next, we observe that (1.2.14) holds for $a = b = 0$ and $z, w \notin \Lambda$. Indeed, both sides are equal to $2\pi i \exp(-A^{-1}z(w - \bar{w})) \cdot F(z, -w, \tau)$, by (1.1.6) and (1.2.5).

Hence, still for $z, w \notin \Lambda$, by (1.2.13) and (1.1.4), we have for any $a, b \geq 0$,

$$\begin{aligned} g_{a,b}^*(z, w) &= \partial_z^b(-\partial_w - A^{-1}\bar{z})^a g_{0,0}^*(z, w) = -\partial_z^b(-\partial_w - A^{-1}\bar{z})^a e_{0,1}^*(z, -w) = \\ &= \frac{(-1)^{a+b+1}b!}{A^a} e_{a,b+1}^*(z, -w), \end{aligned}$$

so we checked (1.2.14) in this case.

Combining (1.1.2) and (1.2.10), we can pass to the limit $z \rightarrow 0$ (by eliminating the singular terms), and so (1.2.14) still holds for $z = 0$ (equivalently, for $z \in \Lambda$) and $w \notin \Lambda$. By (1.1.3)

and (1.2.3), this implies that (1.2.14) also holds when $w \in \Lambda$, $z \notin \Lambda$. Thus, we have covered all the cases. \square

The following result is equivalent (using Theorem C) to [1, Thm. 1.17].

Proposition 1.2.2. *One has for all $z_0, w_0 \in \mathbb{C}$ the following Laurent series expansion of the meromorphic function in z, w at $(0, 0)$:*

$$(1.2.15) \quad \Theta_{z_0, w_0}(z, w) = - \sum_{a, b \geq 0} g_{a, b}^*(z_0, -w_0) \frac{w^a z^b}{a! b!} + \delta_\Lambda(w_0) \frac{1}{w} + \delta_\Lambda(z_0) \frac{\langle w_0, z_0 \rangle}{z},$$

where δ_Λ is the delta-function of the lattice Λ .

Proof. Let us set $G(z, w) = -\Theta_{z_0, w_0}(z, w)$ for brevity. Note that

$$G(z, w) = \exp\left(\frac{(z + z_0)\bar{w} - w\bar{z}_0}{A}\right) g_{0, 0}^*(z_0 + z, -w_0 - w).$$

Assume first that $z_0, w_0 \notin \Lambda$. The fact that G is holomorphic in z, w near $(0, 0)$ follows immediately from Lemma 1.2.1. Now using (1.2.13) we obtain

$$\begin{aligned} \partial_z^b \partial_w^a G|_{z=w=0} &= \exp\left(\frac{(z + z_0)\bar{w} - w\bar{z}_0}{A}\right) \left(\partial_z + \frac{\bar{w}}{A}\right)^b \left(-\partial_w - \frac{\bar{z}_0}{A}\right)^a g_{0, 0}^*(z_0 + z, -w_0 - w)|_{z=w=0} \\ &= g_{a, b}^*(z_0, -w_0), \end{aligned}$$

which gives the required expansion.

Let us now consider the case $z_0 = 0$, $w_0 \notin \Lambda$. In this case using the relation between $g_{0, 0}^*$ and the Kronecker function F (see (1.2.5)) we see that $G(z, w) + z^{-1}$ is holomorphic near $(0, 0)$. Hence, $\partial_z^b \partial_w^a (G + z^{-1})$ is also holomorphic near $(0, 0)$, and we just need to calculate $\partial_z^b \partial_w^a (G + z^{-1})|_{z=w=0}$. Using (1.2.7) and (1.2.9) we obtain

$$\begin{aligned} \partial_z^b \partial_w^a (G + z^{-1})|_{z=w=0} &= \left[\exp\left(\frac{z\bar{w}}{A}\right) \left(\partial_z + \frac{\bar{w}}{A}\right)^b (-\partial_w)^a g_{0, 0}^*(z, -w_0 - w) + \delta_{a, 0} \frac{(-1)^b b!}{z^{b+1}} \right] |_{z=w=0} = \\ &= \left[\exp\left(\frac{z\bar{w}}{A}\right) (-1)^a \sum_{i=0}^a \binom{a}{i} (A^{-1} \bar{z})^i g_{a-i, b}^*(z, -w_0 - w) + \delta_{a, 0} \frac{(-1)^b b!}{z^{b+1}} \right] |_{z=w=0} = g_{a, b}^*(0, -w_0), \end{aligned}$$

where the last equality follows from (1.2.12). The cases $z_0 \notin \Lambda$, $w_0 = 0$ and $z_0 = w_0 = 0$ can be considered similarly. \square

2. CALCULATION OF THE A_∞ -ALGEBRA ASSOCIATED WITH AN ELLIPTIC CURVE

2.1. Homological perturbation construction. The homological perturbation gives an A_∞ -structure on the cohomology of a dg-algebra (A, d) equipped with a projector $\Pi : A \rightarrow B$ onto a subspace of $\ker(d)$ and a homotopy operator Q such that $\text{id} - \Pi = dQ + Qd$. Namely, the formula for the higher products on B is given by (see [5], [7]) as the following sum over trees:

$$m_n(b_1, \dots, b_n) = - \sum_T \epsilon(T) m_T(b_1, \dots, b_n).$$

Here T runs over all oriented planar rooted 3-valent trees with n leaves marked by b_1, \dots, b_n . The expression $m_T(b_1, \dots, b_n)$ is obtained by going down from leaves to the root, applying

the multiplication in A at every vertex, applying the operator Q at every inner edge, and applying Π at the root (see [5, sec. 6.4] for details). The sign $\epsilon(T)$ is given by

$$\epsilon(T) = \prod_v (-1)^{|e_1(v)| + (|e_2(v)| - 1) \deg(e_1(v))},$$

where v runs through vertices of T , $(e_1(v), e_2(v))$ is the pair of edges above v , for an edge e we denote by $|e|$ the total number of leaves above e and by $\deg(e)$ the sum of degrees of markings of all the leaves above e .

In our case the algebra

$$E = \text{Ext}^*(G, G), \text{ where } G = \bigoplus_{i=1}^r P_i \oplus \bigoplus_{j=1}^s L_j,$$

is obtained as the cohomology of the Dolbeault dg-algebra

$$A = (\Omega^{0,*}(\mathcal{E}nd(G)), \bar{\partial}).$$

To construct the homotopy operator Q on A , as in [8], [11], we use the flat metric on C and on the relevant line bundles. Namely, the hermitian metric on L given by

$$(f, g) = \int_C f(z) \overline{g(z)} \exp\left(-2\pi \frac{y^2}{\text{Im}(\tau)}\right) dx dy,$$

where $z = x + iy$. To get metrics on L_j we use the translation $t_{z_j}^*$. Also, we get the induced metrics on $P_i = t_{w_i}^* L \otimes L^{-1}$.

Now A is the direct sum of Dolbeault complexes of the line bundles of the form $L_j \otimes P_i^{-1}$, $P_i \otimes L_j^{-1}$, $L_{j'} \otimes L_j^{-1}$ and $P_{i'} \otimes P_i^{-1}$, and for each such line bundle M we define the homotopy to be the operator

$$Q_M = \bar{\partial}^* G : \Omega^{0,1}(M) \rightarrow \Omega^{0,0}(M),$$

where G is the Green operator corresponding to the Laplacian $\bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$. Then $\Pi = \text{id} - Q_M \bar{\partial} - \bar{\partial} Q_M$ is the orthogonal projection onto the subspace of harmonic forms in A .

The elements of our basis in $E = \text{Ext}^*(G, G)$ have natural harmonic representatives which we denote in the same way. We only need to explain what they are for elements of degree 1: for ξ the representative is $d\bar{z}$, while for η_{ij} we take the translation $t_{z_j - w_i}^*$ of

$$\eta := \sqrt{2 \text{Im}(\tau)} \cdot \overline{\theta(z, \tau)} \exp\left(-2\pi \frac{\text{Im}(z)^2}{\text{Im}(\tau)}\right) d\bar{z},$$

which is a $(0, 1)$ -form with values in L^{-1} .

We also have a natural symmetric bilinear pairing on $A = A^0 \oplus A^1$ given by

$$\langle \alpha, \beta \rangle = \frac{1}{2i \text{Im}(\tau)} \cdot \int_C \text{Tr}(\alpha \circ \beta) \wedge dz,$$

where α and β are homogeneous elements such that $\deg(\alpha) + \deg(\beta) = 1$, which induces a perfect pairing between E^0 and E^1 . By [8, Thm. 1.1], the A_∞ -structure on E satisfies the following cyclic symmetry:

$$(2.1.1) \quad \langle m_n(\alpha_1, \dots, \alpha_n), \alpha_{n+1} \rangle = (-1)^{n(\deg(\alpha_1)+1)} \langle m_n(\alpha_2, \dots, \alpha_{n+1}), \alpha_1 \rangle.$$

The only nonzero double products on E , aside from those involving identity elements, are

$$(2.1.2) \quad m_2(\theta_{ij}, \eta_{ji}) = \xi_i, \quad m_2(\eta_{ji}, \theta_{ij}) = \xi_j.$$

By [11, Lem. 2.1.1], every higher product m_n containing id_{P_i} or id_{L_j} vanishes. Together with the cyclic symmetry (2.1.1) this implies that the only potentially nonzero higher products are of the following types:

- (I) $m_n((\xi_i)^a, \theta_{ij}, (\xi_j)^b, \eta_{ji}, (\xi_i)^c, \theta_{i'j'}, (\xi_{j'})^d) \in \text{Hom}(P_i, L_{j'}),$
- (II) $m_n((\xi_j)^a, \eta_{ji}, (\xi_i)^b, \theta_{ij}, (\xi_{j'})^c, \eta_{j'i'}, (\xi_{i'})^d) \in \text{Ext}^1(L_j, P_{i'}),$
- (III) $m_n((\xi_i)^a, \theta_{ij}, (\xi_j)^b, \eta_{ji}, (\xi_{i'})^c, \theta_{i'j'}, (\xi_{j'})^d, \eta_{j'i}, (\xi_i)^e) \in \text{Hom}(P_i, P_i),$
- (IV) $m_n((\xi_j)^a, \eta_{ji}, (\xi_i)^b, \theta_{ij}, (\xi_{j'})^c, \eta_{j'i'}, (\xi_{i'})^d, \theta_{i'j}, (\xi_j)^e) \in \text{Hom}(L_j, L_j),$

where we denote by $(\xi)^a$ the string (ξ, \dots, ξ) with ξ repeated a times, and the indices in the pairs $(i, i'), (j, j')$ are not necessarily distinct.

Let us set

$$M(a, b, c, d)_{ii'}^{jj'} := \langle m_n((\xi_i)^a, \theta_{ij}, (\xi_j)^b, \eta_{ji}, (\xi_{i'})^c, \theta_{i'j'}, (\xi_{j'})^d), \eta_{j'i} \rangle.$$

Lemma 2.1.1. *The only potentially nonzero products m_n , with $n \geq 3$, of the A_∞ -structure on E are of the form*

$$\begin{aligned} m_n((\xi_i)^a, \theta_{ij}, (\xi_j)^b, \eta_{ji}, (\xi_{i'})^c, \theta_{i'j'}, (\xi_{j'})^d) &= M(a, b, c, d)_{ii'}^{jj'} \cdot \theta_{ij'}, \\ m_n((\xi_j)^a, \eta_{ji}, (\xi_i)^b, \theta_{ij}, (\xi_{j'})^c, \eta_{j'i'}, (\xi_{i'})^d) &= M(b, c, d, a)_{ii'}^{j'j} \cdot \eta_{ji'}, \\ m_n((\xi_i)^a, \theta_{ij}, (\xi_j)^b, \eta_{ji}, (\xi_{i'})^c, \theta_{i'j'}, (\xi_{j'})^d, \eta_{j'i}, (\xi_i)^e) &= M(a + e + 1, b, c, d)_{ii'}^{jj'} \cdot \text{id}_{P_i}, \\ m_n((\xi_j)^a, \eta_{ji}, (\xi_i)^b, \theta_{ij}, (\xi_{j'})^c, \eta_{j'i'}, (\xi_{i'})^d, \theta_{i'j}, (\xi_j)^e) &= M(b, c, d, a + e + 1)_{ii'}^{j'j} \cdot \text{id}_{L_j}. \end{aligned}$$

Proof. This follows immediately from the cyclic symmetry (2.1.1). \square

Thus, it is enough to compute the products of type (I), i.e., the coefficients

$$\langle m_n((\xi_i)^a, \theta_{ij}, (\xi_j)^b, \eta_{ji}, (\xi_{i'})^c, \theta_{i'j'}, (\xi_{j'})^d), \eta_{j'i} \rangle.$$

2.2. Calculation. For a holomorphic line bundle M with a hermitian metric we consider the operator

$$H_M : C^\infty(M) \rightarrow C^\infty(M) : s \mapsto Q_M(s \cdot d\bar{z}),$$

where $Q_M = \bar{\partial}^* G$. Below we will denote all ξ_i and ξ_j simply by ξ . Let us also set for $z \in \mathbb{C}/\Lambda$,

$$L_z := t_z^* L, \quad \theta_z := t_z^* \theta, \quad \eta_z = t_z^* \eta, \quad P_z := t_z^* L \otimes L^{-1}.$$

Note that $L_j \otimes P_i^{-1} = L_{z_j - w_i}$, $P_{i'} \otimes P_i^{-1} = P_{w_{i'} - w_i}$, and $L_{j'} \otimes L_j^{-1} = P_{z_{j'} - z_j}$.

Lemma 2.2.1. *One has*

$$\begin{aligned} \langle m_n((\xi)^a, \theta_{ij}, (\xi)^b, \eta_{ji}, (\xi)^c, \theta_{i'j'}, (\xi)^d), \eta_{j'i} \rangle &= \\ (-1)^{\binom{n}{2}+1} \cdot \sum_{a=a_1+a_2; c=c_1+c_2} \binom{a_2+b}{a_2} \binom{a_1+c_1}{a_1} \binom{c_2+d}{c_2} \Phi_{w_{i'}-w_i, z_{j'}-z_j}(a_2+b, a_1+c_1, c_2+d) &+ \\ (-1)^{\binom{n}{2}+n+1} \cdot \sum_{b=b_1+b_2; d=d_1+d_2} \binom{c+d_1}{c} \binom{b_2+d_2}{b_2} \binom{a+b_1}{a} \Phi_{z_{j'}-z_j, w_{i'}-w_i}(c+d_1, b_2+d_2, a+b_1), & \end{aligned}$$

where for $z_0, z'_0 \in \mathbb{C}$ we set

$$\Phi_{z_0, z'_0}(k, l, p) = (-1)^p \langle \Pi \left([H_{P_{z_0}}^l Q_{P_{z_0}}(H_{L_{z_0}}^k(\theta_{z_0}) \cdot \eta)] \cdot H_{L_{z'_0}}^p(\theta_{z'_0}) \right), \eta_{z_0+z'_0} \rangle.$$

Proof. The proof is similar to that in [11, Sec. 2.3]. \square

We start by calculating the $(0, 1)$ -forms $H_{L_{z_0}}^k(\theta_{z_0})\eta_{z'_0}$ with values in $P_{z_0-z'_0}$. We use real coordinates u, v on \mathbb{C} such that $z = u + v\tau$. Let us also set $a = \text{Im}(\tau)$.

Note that the function $\exp(-2\pi i z_0 v)$ descends to an everywhere nonvanishing (but non-holomorphic) section of the line bundle P_{z_0} . Hence, we have an orthonormal basis of C^∞ -sections of P_{z_0} given by

$$(2.2.1) \quad \varphi_{z_0, m, n}(z) = \varphi_{z_0, m\tau - n}(z) := \exp(2\pi i(mu + nv - z_0 v)) = \langle m\tau - n, z \rangle \cdot \exp(-2\pi i z_0 v).$$

Lemma 2.2.2. *Let us set for $\lambda = m\tau - n$,*

$$c_\lambda(z) = (-1)^{mn} \exp\left(-\frac{\pi}{2a}(|\lambda|^2 + 2\bar{\lambda}z + z^2)\right).$$

For $k \geq 0$ and $z_0, z'_0 \in \mathbb{C}/\Lambda$ one has:

$$(2.2.2) \quad H_{L_{z_0}}^k(\theta_{z_0})\eta_{z'_0} = \frac{1}{k!} \cdot \exp(2\pi i(z'_0 - z_0)v'_0) \cdot \sum_{\lambda \in \Lambda} (\bar{\lambda} - \bar{z}'_0 + \bar{z}_0)^k c_\lambda(z_0 - z'_0) \cdot \langle \lambda, z'_0 \rangle \cdot \varphi_{z_0 - z'_0, \lambda}(z) d\bar{z},$$

where $z'_0 = u'_0 + v'_0\tau$.

Proof. Note that

$$H_{L_{z_0}}^k(\theta_{z_0})\eta_{z'_0} = t_{z_0}^*(H^k(\theta) \cdot t_{z'_0 - z_0}^*\eta),$$

so our identity is equivalent to

$$H^k(\theta)t_{z'_0 - z_0}^*\eta = \frac{1}{k!} \cdot \exp(2\pi i(z'_0 - z_0)(v'_0 - v_0)) \cdot \sum_{\lambda} (\bar{\lambda} - \bar{z}'_0 + \bar{z}_0)^k c_\lambda(z_0 - z'_0) \cdot \langle \lambda, z'_0 - z_0 \rangle \cdot \varphi_{z_0 - z'_0, \lambda}(z) \cdot d\bar{z},$$

where $z_0 = u_0 + v_0\tau$. Recall that we have

$$H^k(\theta) = \frac{1}{k!} D^k(\theta),$$

where $D = -\frac{a}{\pi} \frac{\partial}{\partial z} + \bar{z} - z$ (see [11, Lem. 2.4.1]), so our assertion is equivalent (upon replacing $z'_0 - z_0$ with z_0) to

$$(2.2.3) \quad \begin{aligned} & \sqrt{2a} \cdot D^k(\theta) \cdot \overline{\theta(z + z_0)} \exp(-2\pi a(v + v_0)^2) = \\ & \exp(2\pi i z_0 v_0) \cdot \sum_{\lambda} (\bar{\lambda} - \bar{z}_0)^k c_\lambda(-z_0) \cdot \langle \lambda, z_0 \rangle \cdot \varphi_{-z_0, \lambda}(z). \end{aligned}$$

For $k = 0$ this follows immediately from [9, eq. (2.2)]. To deduce the general case we use induction on k . Namely, since

$$-\frac{a}{\pi} \frac{\partial}{\partial z} \varphi_{-z_0, \lambda} = (\bar{\lambda} - z_0) \varphi_{-z_0, \lambda}, \quad -\frac{a}{\pi} \frac{\partial}{\partial z} \exp(-2\pi a v^2) = (\bar{z} - z) \exp(-2\pi a v^2),$$

applying D to both sides of the identity (2.2.3) we get

$$\begin{aligned} & \sqrt{2a} \cdot D^{k+1}(\theta) \cdot \overline{\theta(z+z_0)} \exp(-2\pi a(v+v_0)^2) + \\ & (\overline{z+z_0} - z - z_0) \cdot \sqrt{2a} \cdot D^k(\theta) \cdot \overline{\theta(z+z_0)} \exp(-2\pi a(v+v_0)^2) = \exp(2\pi i z_0 v_0) \times \\ & \sum_{\lambda} (\bar{\lambda} - \bar{z}_0)^k (\bar{\lambda} - z_0 + \bar{z} - z) c_{\lambda}(-z_0) \cdot \langle \lambda, z_0 \rangle \cdot \varphi_{-z_0, \lambda}(z). \end{aligned}$$

It remains to use (2.2.3) to replace the second term in the left-hand side, and the identity for $k+1$ follows. \square

Now we can calculate the expressions $\Phi_{z_0, w_0}(k, l, p)$ (see Lemma 2.2.1). Note that the calculations for $z_0 \in \Lambda$ and $z_0 \notin \Lambda$ are slightly different, since the homotopy operator $Q_{P_{z_0}}$ behaves differently when P_{z_0} is trivial.

Lemma 2.2.3. *For either $z_0 = 0$ or $z_0 \notin \Lambda$ one has*

$$\begin{aligned} \Phi_{z_0, w_0}(k, l, p) &= \frac{A^{l+1}}{k!p!} \cdot \exp(A^{-1}z_0(w_0 - \bar{w}_0)) \cdot \sum_{\lambda \in \Lambda \setminus \{-z_0\}} \frac{(\bar{\lambda} + \bar{z}_0)^{k+p}}{(\lambda + z_0)^{l+1}} \exp(-A^{-1}|\lambda + z_0|^2) \langle -\lambda, w_0 \rangle \\ &= \frac{A^{k+p+l+1}}{k!p!} \cdot \tilde{f}_{k+p, l+1}^*(z_0, w_0). \end{aligned}$$

Proof. Assume first that $z_0 \notin \Lambda$. Then P_{z_0} is a nontrivial line bundle of degree 0, so that $H^0(C, L) = H^1(C, L) = 0$. Hence, in this case we have $Q_{P_{z_0}} = \bar{\partial}^{-1}$. For the elements of the basis (2.2.1) we have

$$\begin{aligned} \bar{\partial}\varphi_{z_0, \lambda} &= A^{-1}(\lambda + z_0)\varphi_{z_0, \lambda}d\bar{z}, \\ Q_{P_{z_0}}(\varphi_{z_0, \lambda}d\bar{z}) &= \frac{A}{\lambda + z_0}\varphi_{z_0, \lambda}, \end{aligned}$$

and the formula for $H_{P_{z_0}}$ is similar.

Hence, using Lemma 2.2.2 we obtain

$$Q_{P_{z_0}}(H_{L_{z_0}}^k(\theta_{z_0}) \cdot \eta) = \frac{A}{k!} \cdot \sum_{\lambda} \frac{(\bar{\lambda} + \bar{z}_0)^k}{\lambda + z_0} c_{\lambda}(z_0) \varphi_{z_0, \lambda}(z).$$

Therefore,

$$(2.2.4) \quad H_{P_{z_0}}^l Q_{P_{z_0}}(H_{L_{z_0}}^k(\theta_{z_0}) \cdot \eta) = \frac{A^{l+1}}{k!} \cdot \sum_{\lambda} \frac{(\bar{\lambda} + \bar{z}_0)^k}{(\lambda + z_0)^{l+1}} c_{\lambda}(z_0) \varphi_{z_0, \lambda}(z).$$

Next, comparing the formulas for η and for the metric on L we observe that for a C^∞ -section f of L one has $\langle f, \theta \rangle = 0$ if and only if $\langle f, \eta \rangle = 0$. Hence, for $f \in C^\infty(L)$ one has

$$\langle \Pi(f), \eta \rangle = \langle f, \eta \rangle$$

(since Π is the orthogonal projection onto $\mathbb{C}\theta$). The same property holds for $\eta_w \in \Omega^{0,1}(L_w)$. Therefore,

$$\begin{aligned} \Phi_{z_0, w_0}(k, l, p) &= (-1)^p \langle [H_{P_{z_0}}^l Q_{P_{z_0}}(H_{L_{z_0}}^k(\theta_{z_0}) \cdot \eta)] \cdot H_{L_{w_0}}^p(\theta_{w_0}), \eta_{z_0+w_0} \rangle = \\ &= (-1)^p \langle H_{P_{z_0}}^l Q_{P_{z_0}}(H_{L_{z_0}}^k(\theta_{z_0}) \cdot \eta), H_{L_{w_0}}^p(\theta_{w_0}) \cdot \eta_{z_0+w_0} \rangle. \end{aligned}$$

Now we observe that

$$\langle \varphi_{z_0, \lambda}, \varphi_{-z_0, \lambda'} \rangle = \delta_{\lambda + \lambda', 0}.$$

Hence, the answer can be computed using the Fourier expansion for $H_{L_{w_0}}^p(\theta_{w_0}) \cdot \eta_{z_0 + w_0}$ from Lemma 2.2.2 and the Fourier expansion (2.2.4):

$$\begin{aligned} \Phi_{z_0, w_0}(k, l, p) &= \frac{A^{l+1}}{k!p!} \cdot \exp(2\pi i z_0(v_0 + v'_0)) \sum_{\lambda} \frac{(\bar{\lambda} + \bar{z}_0)^{k+p}}{(\lambda + z_0)^{l+1}} c_{\lambda}(z_0)^2 \langle -\lambda, z_0 + w_0 \rangle = \\ &= \frac{A^{l+1}}{k!p!} \cdot \exp(2\pi i z_0 v'_0) \cdot \sum_{\lambda} \frac{(\bar{\lambda} + \bar{z}_0)^{k+p}}{(\lambda + z_0)^{l+1}} \exp(-A^{-1}|\lambda + z_0|^2) \langle -\lambda, w_0 \rangle, \end{aligned}$$

where $z_0 = u_0 + v_0\tau$, $w_0 = u'_0 + v'_0\tau$.

In the case $z_0 \in \Lambda$ the calculation is similar except that we use the fact that the operator $Q_{\mathcal{O}} : \Omega^{0,1} \rightarrow \Omega^{0,0}$ is given by

$$Q_{\mathcal{O}}(\varphi_{0,m,n}(z)d\bar{z}) = \begin{cases} \frac{a}{\pi(m\tau - n)} \varphi_{0,m,n}(z), & (m, n) \neq (0, 0), \\ 0 & (m, n) = (0, 0). \end{cases}$$

□

Proof of Theorem A. Substituting the expression for $\Phi_{z_0, w_0}(k, l, p)$ into the formula of Lemma 2.2.1 we get

$$\begin{aligned} &\langle m_n((\xi)^a, \theta_{ij}, (\xi)^b, \eta_{ji'}, (\xi)^c, \theta_{i'j'}, (\xi)^d), \eta_{j'i} \rangle = \\ &(-1)^{\binom{n}{2}+1} A^{n-2} \cdot \sum_{a=a_1+a_2; c=c_1+c_2} \frac{(a_1 + c_1)!}{a_1!a_2!c_1!c_2!b!d!} \tilde{f}_{a_2+c_2+b+d, a_1+c_1+1}^*(w_{i'} - w_i, z_{j'} - z_j) + \\ &(-1)^{\binom{n}{2}+n+1} A^{n-2} \cdot \sum_{b=b_1+b_2; d=d_1+d_2} \frac{(b_2 + d_2)!}{b_1!b_2!d_1!d_2!a!c!} \tilde{f}_{b_1+d_1+a+c, b_2+d_2+1}^*(z_{j'} - z_j, w_{i'} - w_i). \end{aligned}$$

Denoting $k = a_1 + c_1$ and $l = b_2 + d_2$ we can rewrite this as

$$\begin{aligned} &(-1)^{\binom{n}{2}+1} A^{-n+2} \cdot \langle m_n((\xi)^a, \theta_{ij}, (\xi)^b, \eta_{ji'}, (\xi)^c, \theta_{i'j'}, (\xi)^d), \eta_{j'i} \rangle = \\ &\frac{1}{b!d!} \cdot \sum_{k \geq 0} C(a, c, k) \tilde{f}_{a+c-k+b+d, k+1}^*(w_{i'} - w_i, z_{j'} - z_j) + \\ &(-1)^n \frac{1}{a!c!} \cdot \sum_{l \geq 0} C(b, d, l) \tilde{f}_{b+d-l+a+c, l+1}^*(z_{j'} - z_j, w_{i'} - w_i), \end{aligned}$$

where

$$C(a, c, k) = \sum_{a_1+c_1=k; a_1 \leq a, c_1 \leq c} \frac{k!}{a_1!c_1!(a-a_1)!(c-c_1)!} = \frac{k!}{a!c!} \sum_{a_1+c_1=k} \binom{a}{a_1} \binom{c}{c_1} = \frac{k!}{a!c!} \binom{a+c}{k}$$

Thus, our formula takes form

$$\begin{aligned} &(-1)^{\binom{n}{2}+1} A^{-n+2} \cdot \langle m_n((\xi)^a, \theta_{ij}, (\xi)^b, \eta_{ji'}, (\xi)^c, \theta_{i'j'}, (\xi)^d), \eta_{j'i} \rangle = \frac{1}{a!b!c!d!} \times \\ &\sum_{k \geq 0} k! \left(\binom{a+c}{k} \tilde{f}_{n-3-k, k+1}^*(w_{i'} - w_i, z_{j'} - z_j) + (-1)^n \binom{b+d}{k} \tilde{f}_{n-3-k, k+1}^*(z_{j'} - z_j, w_{i'} - w_i) \right). \end{aligned}$$

Comparing with (1.2.1) we obtain

$$(2.2.5) \quad M(a, b, c, d)_{ii'}^{jj'} = \langle m_n((\xi)^a, \theta_{ij}, (\xi)^b, \eta_{ji'}, (\xi)^c, \theta_{i'j'}, (\xi)^d), \eta_{ji'} \rangle = \\ (-1)^{\binom{n}{2}+1} \frac{A^{n-2}}{a!b!c!d!} \cdot \tilde{g}_{a+c,b+d}^*(z_{j'} - z_j, w_{i'} - w_i),$$

where $n = a + b + c + d + 3$. Taking into account Theorem C we can rewrite this in terms of the Eisenstein-Kronecker numbers as

$$m_n((\xi_i)^a, \theta_{ij}, (\xi_j)^b, \eta_{ji'}, (\xi_{i'})^c, \theta_{i'j'}, (\xi_{j'})^d) = (-1)^{\binom{n+1}{2}+1} \frac{A^{b+d+1} \cdot (b+d)!}{a!b!c!d!} \times \\ \exp(A^{-1}(z_{j'} - z_j)(w_{i'} - w_i - \overline{w_{i'}} + \overline{w_i})) \cdot e_{a+c,b+d+1}^*(z_{j'} - z_j, w_i - w_{i'}) \cdot \theta_{ij'},$$

which is equivalent to the formula of Theorem A. The remaining A_∞ -products are determined by this, as follows from Lemma 2.1.1. \square

2.3. Consequences of the A_∞ -constraint. Now we will determine the identities for $g_{a,b}^*(z, w)$ (or equivalently, for $e_{a,b+1}^*(z, w)$) obtained from the A_∞ -identities on the structure constants $M(a, b, c, d)_{ii'}^{jj'}$.

Proposition 2.3.1. *For any integers $a, b \geq 0$ one has, for $z, z', w, w' \in \mathbb{C}$,*

$$\sum_{a=a_1+a_2} \binom{a}{a_1} g_{a_2,0}^*(z, w) g_{a_1,b}^*(z', w + w') - \sum_{b=b_1+b_2} \binom{b}{b_1} g_{0,b_1}^*(z', w') g_{a,b_2}^*(z + z', w) + \\ g_{0,0}^*(-z, w') g_{a,b}^*(z + z', w + w') = \delta_{b,0} \delta_\Lambda(w + w') \frac{1}{a+1} g_{a+1,0}^*(z, w) \\ - \delta_{a,0} \delta_\Lambda(z + z') \frac{\langle z + z', w \rangle}{b+1} g_{0,b+1}^*(z', w') + \delta_\Lambda(w) \frac{1}{a+1} g_{a+1,b}^*(z', w') \\ - \delta_\Lambda(z) \langle z, w \rangle g_{a,b+1}^*(z', w + w') + \delta_\Lambda(w') g_{a+1,b}^*(z + z', w) - \delta_\Lambda(z') \frac{\langle z', w + w' \rangle}{b+1} g_{a,b+1}^*(z, w).$$

Proof. Applying the A_∞ -axiom to the string

$$(\xi_i)^a, \theta_{ij}, \eta_{ji'}, \theta_{i'j'}, \eta_{ji''}, \theta_{i''j''}, (\xi_{j''})^b$$

we get

$$\sum_{a=a_1+a_2} (-1)^{(a_2+1)(a_1+b)+a_1} M(a_2, 0, 0, 0)_{ii'}^{jj'} M(a_1, 0, 0, b)_{ii''}^{j'j''} + \\ \sum_{b=b_1+b_2} (-1)^{(b_1+1)(a+b_2+1)+a} M(0, 0, 0, b_1)_{i'i''}^{j'j''} M(a, 0, 0, b_2)_{ii'}^{j'j''} + \\ (-1)^b M(0, 0, 0, 0)_{i'i''}^{j'j} M(a, 0, 0, b)_{ii''}^{j'j''} + (-1)^a \delta_{b,0} \delta_{i,i''} M(a+1, 0, 0, 0)_{ii'}^{j'j'} \\ - \delta_{a,0} \delta_{j,j''} M(0, 0, 0, b+1)_{i'i''}^{j'j''} + (-1)^a [\delta_{i,i'} M(a+1, 0, 0, b)_{i'i''}^{j'j''} - \delta_{j,j'} M(a, 1, 0, b)_{ii''}^{j'j''} + \\ \delta_{i',i''} M(a, 0, 1, b)_{ii'}^{j'j''} - \delta_{j',j''} M(a, 0, 0, b+1)_{ii'}^{j'j'}] = 0.$$

Using (2.2.5) we get the required identity for $z = z_{j'} - z_j$, $z' = z_{j''} - z_{j'}$, $w = w_{i'} - w_i$, $w' = w_{i''} - w_{i'}$. Recall that by our choice of (w_i) , (z_j) , each of these differences is either 0 or in $\mathbb{C} \setminus \Lambda$. The general case of the identity follows using (1.2.2). \square

Specializing the variables in Proposition 2.3.1 by setting $w' = z' = 0$, or $w = z = 0$, or $w = w' = z' = 0$, or $w = w' = z = 0$, we get the following identities.

Corollary 2.3.2. *For $z, w \notin \Lambda$ one has*

(2.3.1)

$$g_{a+1,b}^*(z, w) - \frac{1}{b+1} g_{a,b+1}^*(z, w) = \sum_{a=a_1+a_2} \binom{a}{a_1} g_{a_2,0}^*(z, w) g_{a_1,b}^*(0, w) - \sum_{b=b_1+b_2} \binom{b}{b_1} g_{0,b_1}^*(0, 0) g_{a,b_2}^*(z, w) + g_{0,0}^*(-z, 0) g_{a,b}^*(z, w);$$

(2.3.2)

$$\frac{1}{a+1} g_{a+1,b}^*(z, w) - g_{a,b+1}^*(z, w) = \sum_{a=a_1+a_2} \binom{a}{a_1} g_{a_2,0}^*(0, 0) g_{a_1,b}^*(z, w) - \sum_{b=b_1+b_2} \binom{b}{b_1} g_{0,b_1}^*(z, w) g_{a,b_2}^*(z, 0) + g_{0,0}^*(0, w) g_{a,b}^*(z, w);$$

For $z \notin \Lambda$ one has

$$(2.3.3) \quad \left(1 + \frac{\delta_{b,0}}{a+1}\right) g_{a+1,b}^*(z, 0) - \frac{1}{b+1} g_{a,b+1}^*(z, 0) = \sum_{a=a_1+a_2} \binom{a}{a_1} g_{a_2,0}^*(z, 0) g_{a_1,b}^*(0, 0) - \sum_{b=b_1+b_2} \binom{b}{b_1} g_{0,b_1}^*(0, 0) g_{a,b_2}^*(z, 0) + g_{0,0}^*(-z, 0) g_{a,b}^*(z, 0) - \frac{1}{a+1} g_{a+1,0}^*(0, 0)$$

$$(2.3.4) \quad \left(1 + \frac{1}{a+1}\right) g_{a+1,b}^*(z, 0) - g_{a,b+1}^*(z, 0) = \sum_{a=a_1+a_2} \binom{a}{a_1} g_{a_2,0}^*(0, 0) g_{a_1,b}^*(z, 0) - \sum_{b=b_1+b_2} \binom{b}{b_1} g_{0,b_1}^*(z, 0) g_{a,b_2}^*(z, 0) + -\frac{\delta_{b,0}}{a+1} g_{a+1,0}^*(0, 0).$$

Note that for $a = 0$ and $b \geq 1$ the equation (2.3.4) becomes

$$2g_{1,b}^*(z, 0) = g_{0,b+1}^*(z, 0) - \sum_{b=b_1+b_2} \binom{b}{b_1} g_{0,b_1}^*(z, 0) g_{0,b_2}^*(z, 0),$$

which is equivalent to [15, VI.4, Eq. (10)].

Proof of Corollary B. The fact that all $A^{-a} e_{a,b}^*(0, 0)$ can be expressed as polynomials with rational coefficients in Eisenstein series (e_n^*) is proved in [15, VI.5] (see also [11, Prop. 2.6.1]). It is also well known that e_n with $n \geq 8$ are expressed in terms of e_4 and e_6 . Thus, by (1.2.14), the assertion reduces to checking that for some polynomial P with rational coefficients one has

$$g_{a,b}^*(z, w) = P(g_{0,0}^*(z, w), g_{0,1}^*(z, w), g_{0,0}^*(z), g_{0,1}^*(z), g_{0,2}^*(z), g_{0,0}^*(w), g_{0,1}^*(w), g_{0,2}^*(w), (g_{m,n}^*)),$$

where we set $g_{a,b}^*(z) := g_{a,b}^*(z, 0)$, $g_{a,b}^* := g_{a,b}^*(0, 0)$.

First, let us prove the similar assertion for $w = 0$, $z \notin \Lambda$. In other words, we claim that there is a polynomial P with rational coefficients such that

$$g_{a,b}^*(z) = P(g_{0,0}^*(z), g_{0,1}^*(z), g_{0,2}^*(z), (g_{m,n}^*)),$$

The induction on $a + b$, using identities (2.3.3) and (2.3.4), shows that we can express any $g_{a,b}^*(z)$ in terms of $g_{a',b'}^*(z)$ with $a' + b' \leq 2$ and $(g_{m,n}^*)$. Furthermore, using (2.3.4), we can express these in terms of just $g_{0,0}^*(z)$, $g_{0,1}^*(z)$, $g_{0,2}^*(z)$ and $g_{m,n}^*$, as required.

Now in the case $z, w \notin \Lambda$ we similarly use (2.3.1) and (2.3.2) (and induction on $a + b$) to express $g_{a,b}^*(z, w)$ in terms of $g_{0,0}^*(z, w)$, $g_{0,1}^*(z, w)$ and $(g_{m,n}^*(z, 0))$. \square

Remark. For $a = b = 0$ and generic z, w, z', w' the identity of Proposition 2.3.1 becomes

$$g_{0,0}^*(z, w) g_{0,0}^*(z', w + w') - g_{0,0}^*(z', w') g_{0,0}^*(z + z', w) + g_{0,0}^*(-z, w') g_{0,0}^*(z + z', w + w') = 0$$

which means that $g_{0,0}^*$ is a scalar solution of the *associative Yang-Baxter equation (AYBE)*. This is equivalent to the fact that the Kronecker function $F(z, w)$ is a solution of the AYBE (see [10]).

2.4. Variation of the A_∞ -structure with respect to the parameters. Let us fix the lattice Λ . We can view our collection of A_∞ -structures on $E = \text{Ext}^*(G, G)$ depending on the parameters (w_i) , (z_j) , as a bundle of minimal A_∞ -algebras on an open subset of \mathbb{C}^{r+s} (recall that we require $w_{i'} - w_i \notin \Lambda$ for $i \neq i'$ and $z_{j'} - z_j \notin \Lambda$ for $j' \neq j$). The vector bundle is trivial with the fiber E . Since we want to differentiate along the parameters on the base of the family, the choice of a trivialization of this bundle is important for us. We are going to use the basis (0.0.1) in E to define such a trivialization. Note that with respect to this basis the product m_2 is constant in our family.

Recall that for a graded K -module A , where K is a commutative ring, there is a graded Lie algebra on the space of Hochschild cochains given by the Gerstenhaber bracket. For $f \in \text{Hom}_K(A^{\otimes m}, A)$ and $g \in \text{Hom}_K(A^{\otimes n}, A)$, homogeneous of some degree with respect to the grading on A , this bracket is defined as follows:

$$[f, g] = f \circ g - (-1)^{|f||g|} g \circ f, \quad \text{where } f \circ g(a_1, \dots, a_{m+n-1}) = \sum_{i=1}^m (-1)^{(|a_1| + \dots + |a_{i-1}| + m - 1) \deg(g) + (i-1)(n-1)} f(a_1, \dots, a_{i-1}, g(a_i, \dots, a_{i+n-1}), a_{i+n}, \dots, a_{m+n-1}),$$

where for a cochain $f \in \text{Hom}_K(A^{\otimes m}, A)$, homogeneous of degree $\deg(f)$, we set $|f| = \deg(f) + m - 1$. Note that this bracket corresponds to the supercommutator under the standard identification of the Hochschild cochains with coderivations of the free coalgebra cogenerated by $A[1]$. In particular, it is a graded Lie bracket with respect to the grading $|f|$.

A collection $m = (m_n)$, where $m_n \in \text{Hom}_K(A^{\otimes n}, A)$, $\deg(m_n) = 2 - n$, defines an A_∞ -algebra over K if and only if $[m, m] = 0$. The Hochschild cohomology of an A_∞ -algebra A is defined as the differential of the operator $[m, ?]$ on Hochschild cochains.

In our case E has a natural structure of K -algebra for $K = \mathbb{C}^{r+s}$ (this corresponds to the idempotents in E given by the identity morphisms of P_i and L_j).

Proposition 2.4.1. *For each $j = 1, \dots, s$, $i = 1, \dots, r$, let us consider the Hochschild cochains $f_0(j)$, $f_0(i)$, $f_1(i)$, $f'_1(i)$, $f_2(j)$ and $f_2(i)$, where $f_a(*) \in \text{Hom}_K(E^{\otimes a}, E)$ (depending on the parameters (z_j)), given by*

$$\begin{aligned} f_0(j) &= \tilde{\xi}_j, \quad f_0(i) = -\tilde{\xi}_i, \\ f_1(i)(\tilde{\theta}_{ij}) &= -\overline{z_j} \tilde{\theta}_{ij}, \quad f_1(i)(\tilde{\eta}_{ji}) = \overline{z_j} \tilde{\eta}_{ji}, \quad f'_1(i)(\tilde{\theta}_{ij}) = z_j \tilde{\theta}_{ij}, \quad f'_1(i)(\tilde{\eta}_{ji}) = -z_j \tilde{\eta}_{ji}, \\ -f_2(j)(\tilde{\xi}_i, \tilde{\theta}_{ij}) &= f_2(i)(\tilde{\theta}_{ij}, \tilde{\xi}_j) = \tilde{\theta}_{ij}, \quad f_2(j)(\tilde{\eta}_{ji}, \tilde{\xi}_i) = -f_2(i)(\tilde{\xi}_j, \tilde{\eta}_{ji}) = \tilde{\eta}_{ji}, \end{aligned}$$

with all the other components of these cochains being zero. With respect to the trivialization of E given by the basis (0.0.1) (along with the identity elements) one has over the open domain where $w_{i'} - w_i \notin \Lambda$ for $i \neq i'$ and $z_{j'} - z_j \notin \Lambda$ for $j' \neq j$,

$$(2.4.1) \quad \begin{aligned} \partial_{z_j} m_n &= [m_{n+1}, f_0(j)], \quad \partial_{w_i} m_n = [m_{n+1}, f_0(i)] + [m_n, f_1(i)], \\ A \partial_{\overline{z_j}} m_n &= [m_{n-1}, f_2(j)], \quad A \partial_{\overline{w_i}} m_n = [m_{n-1}, f_2(i)] + [m_n, f'_1(i)]. \end{aligned}$$

Let us define a connection ∇ on E by requiring the basis (0.0.1) to be horizontal with respect to ∇_{z_j} and $\nabla_{\bar{z}_j}$, and by setting

$$\begin{aligned}\nabla_{w_i}(\tilde{\theta}_{ij}) &= -A^{-1}\bar{z}_j\tilde{\theta}_{ij}, & \nabla_{w_i}(\tilde{\eta}_{ji}) &= A^{-1}\bar{z}_j\tilde{\eta}_{ji}, \\ \nabla_{\bar{w}_i}(\tilde{\theta}_{ij}) &= A^{-1}z_j\tilde{\theta}_{ij}, & \nabla_{\bar{w}_i}(\tilde{\eta}_{ji}) &= -A^{-1}z_j\tilde{\eta}_{ji},\end{aligned}$$

with other basis elements being horizontal with respect to ∇_{w_i} and $\nabla_{\bar{w}_i}$. Then we can rewrite (2.4.1) as

$$(2.4.2) \quad \begin{aligned}\nabla_{z_j}m_n &= [m_{n+1}, f_0(j)], & \nabla_{w_i}m_n &= [m_{n+1}, f_0(i)], \\ A\nabla_{\bar{z}_j}m_n &= [m_{n-1}, f_2(j)], & A\nabla_{\bar{w}_i}m_n &= [m_{n-1}, f_2(i)].\end{aligned}$$

Proof. We use the formula of Theorem A in the form

$$(2.4.3) \quad m_n((\tilde{\xi}_i)^a, \tilde{\theta}_{ij}, (\tilde{\xi}_j)^b, \tilde{\eta}_{ji'}, (\tilde{\xi}_{i'})^c, \tilde{\theta}_{i'j'}, (\tilde{\xi}_{j'})^d) = \widetilde{M}(a, b, c, d)_{ii'}^{jj'} \cdot \tilde{\theta}_{ij'}, \quad \text{where}$$

$$\widetilde{M}(a, b, c, d)_{ii'}^{jj'} = (-1)^{\binom{n}{2}+1} \frac{1}{a!b!c!d!} \cdot g_{a+c, b+d}^*(z_{j'} - z_j, w_{i'} - w_i),$$

where $n = a + b + c + d + 3$. Now using (1.2.7) and (1.2.8) we get

$$\begin{aligned}\partial_{z_j}\widetilde{M}(a, b, c, d)_{ii'}^{jj'} &= (-1)^{n+1}(b+1)\widetilde{M}(a, b+1, c, d)_{ii'}^{jj'}, \\ \partial_{z_{j'}}\widetilde{M}(a, b, c, d)_{ii'}^{jj'} &= (-1)^n(d+1)\widetilde{M}(a, b, c, d+1)_{ii'}^{jj'}, \\ A\partial_{\bar{z}_j}\widetilde{M}(a, b, c, d) &= (-1)^n[\widetilde{M}(a-1, b, c, d)_{ii'}^{jj'} + \widetilde{M}(a, b, c-1, d)_{ii'}^{jj'}], \\ A\partial_{\bar{z}_{j'}}\widetilde{M}(a, b, c, d) &= (-1)^{n+1}[\widetilde{M}(a-1, b, c, d)_{ii'}^{jj'} + \widetilde{M}(a, b, c-1, d)_{ii'}^{jj'}], \\ \partial_{w_i}\widetilde{M}(a, b, c, d)_{ii'}^{jj'} &= (-1)^{n+1}(a+1)\widetilde{M}(a+1, b, c, d)_{ii'}^{jj'} + A^{-1}(\bar{z}_{j'} - \bar{z}_j)\widetilde{M}(a, b, c, d)_{ii'}^{jj'}, \\ \partial_{w_{i'}}\widetilde{M}(a, b, c, d)_{ii'}^{jj'} &= (-1)^n(c+1)\widetilde{M}(a, b, c+1, d)_{ii'}^{jj'} - A^{-1}(\bar{z}_{j'} - \bar{z}_j)\widetilde{M}(a, b, c, d)_{ii'}^{jj'}, \\ A\partial_{\bar{w}_i}\widetilde{M}(a, b, c, d) &= (-1)^{n+1}[\widetilde{M}(a, b-1, c, d)_{ii'}^{jj'} + \widetilde{M}(a, b, c, d-1)_{ii'}^{jj'}] + (z_{j'} - z_j)\widetilde{M}(a, b, c, d)_{ii'}^{jj'}, \\ A\partial_{\bar{w}_{i'}}\widetilde{M}(a, b, c, d) &= (-1)^n[\widetilde{M}(a, b-1, c, d)_{ii'}^{jj'} + \widetilde{M}(a, b, c, d-1)_{ii'}^{jj'}] - (z_{j'} - z_j)\widetilde{M}(a, b, c, d)_{ii'}^{jj'}.\end{aligned}$$

Note that in the above formulas the terms containing negative arguments in $\widetilde{M}(?, ?, ?, ?)$ should be omitted. Now the assertion follows by direct calculation (note that one should also consider other products appearing in Lemma 2.1.1). For example, let us consider the equation

$$A\nabla_{\bar{w}_i}m_n = [m_{n-1}, f_2(i)]$$

for the products of type (2.4.3). Since $|m_{n-1}| = 2$, the right-hand side is the usual commutator, so

$$\begin{aligned}&[m_{n-1}, f_2(i)]((\tilde{\xi}_i)^a, \tilde{\theta}_{ij}, (\tilde{\xi}_j)^b, \tilde{\eta}_{ji'}, (\tilde{\xi}_{i'})^c, \tilde{\theta}_{i'j'}, (\tilde{\xi}_{j'})^d) = \\ &(-1)^n m_{n-1}((\tilde{\xi}_i)^a, f_2(i)(\tilde{\theta}_{ij}, \tilde{\xi}_j), (\tilde{\xi}_j)^{b-1}, \tilde{\eta}_{ji'}, (\tilde{\xi}_{i'})^c, \tilde{\theta}_{i'j'}, (\tilde{\xi}_{j'})^d) \\ &- (-1)^{n+1} f_2(i)(m_{n-1}((\tilde{\xi}_i)^a, \tilde{\theta}_{ij}, (\tilde{\xi}_j)^b, \tilde{\eta}_{ji'}, (\tilde{\xi}_{i'})^c, \tilde{\theta}_{i'j'}, (\tilde{\xi}_{j'})^{d-1}), \tilde{\xi}_{j'}) = \\ &(-1)^n m_{n-1}((\tilde{\xi}_i)^a, \tilde{\theta}_{ij}, (\tilde{\xi}_j)^{b-1}, \tilde{\eta}_{ji'}, (\tilde{\xi}_{i'})^c, \tilde{\theta}_{i'j'}, (\tilde{\xi}_{j'})^d) + \\ &(-1)^n m_{n-1}((\tilde{\xi}_i)^a, \tilde{\theta}_{ij}, (\tilde{\xi}_j)^b, \tilde{\eta}_{ji'}, (\tilde{\xi}_{i'})^c, \tilde{\theta}_{i'j'}, (\tilde{\xi}_{j'})^{d-1}) = \\ &(-1)^n [\widetilde{M}(a, b-1, c, d)_{ii'}^{jj'} + \widetilde{M}(a, b, c, d-1)_{ii'}^{jj'}] \tilde{\theta}_{ij'}.\end{aligned}$$

On the other hand,

$$(A\nabla_{\overline{w}_i} m_n)((\tilde{\xi}_i)^a, \tilde{\theta}_{ij}, (\tilde{\xi}_j)^b, \tilde{\eta}_{ji'}, (\tilde{\xi}_{i'})^c, \tilde{\theta}_{i'j'}, (\tilde{\xi}_{j'})^d) = A\partial_{\overline{w}_i} \widetilde{M}(a, b, c, d)_{ii'}^{jj'} \tilde{\theta}_{ij'} + \\ (z_{j'} - z_j) \widetilde{M}(a, b, c, d)_{ii'}^{jj'} \tilde{\theta}_{ij'},$$

which matches the above calculation of $[m_{n-1}, f_2(i)]$, due to our formula for $A\partial_{\overline{w}_i} \widetilde{M}(a, b, c, d)_{ii'}^{jj'}$. \square

The first two equations in (2.4.2) can be rewritted as

$$\nabla_{z_j} m_n(x_1, \dots, x_n) = (-1)^n \sum_{k=1}^{n+1} (-1)^{k-1+|x_1|+\dots+|x_{k-1}|} m_{n+1}(x_1, \dots, x_{k-1}, \tilde{\xi}_j, x_k, \dots, x_n), \\ \nabla_{w_i} m_n(x_1, \dots, x_n) = (-1)^{n+1} \sum_{k=1}^{n+1} (-1)^{k-1+|x_1|+\dots+|x_{k-1}|} m_{n+1}(x_1, \dots, x_{k-1}, \tilde{\xi}_i, x_k, \dots, x_n),$$

where (x_i) are elements of the basis (0.0.1). Thus, in view of Theorem A we get a conceptual interpretation of the identity (1.2.13): applications of ∇_{z_j} (resp., ∇_{w_i}) correspond to insertions of $\tilde{\xi}_j$ (resp., $\tilde{\xi}_i$). Note that equations of this form were also obtained by Tu [13] for the family of A_∞ -structures associated with the fibers of a Lagrangian tori fibration.

BIBLIOGRAPHY

- [1] K. Bannai, S. Kobayashi, *Algebraic theta functions and the p -adic interpolation of Eisenstein-Kronecker numbers*, Duke Math. J. 153 (2010), no. 2, 229–295.
- [2] P. Colmez, L. Schneps, *p -adic interpolation of special values of Hecke L -functions*, Compositio Math. 82 (1992), 143–187.
- [3] K. Fukaya, *Cyclic symmetry and adic convergence in Lagrangian Floer theory*, Kyoto J. Math. 50 (2010), 521–590.
- [4] B. Keller, *Introduction to A -infinity algebras and modules*, Homology Homotopy Appl. 3 (2001), no. 1, 1–35.
- [5] M. Kontsevich, Y. Soibelman, *Homological mirror symmetry and torus fibration*, in *Symplectic geometry and mirror symmetry (Seoul, 2000)*, 203–263, World Sci. Publishing, River Edge, NJ, 2001.
- [6] Y. Lekili, A. Polishchuk, *A modular compactification of $\mathcal{M}_{1,n}$ from A_∞ -structures*, arXiv:1408.0611.
- [7] S. Merkulov, *Strong homotopy algebras of a Kähler manifold*, IMRN 1999, no.3, 153–164.
- [8] A. Polishchuk, *Homological mirror symmetry with higher products*, in *Proceedings of the Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds*, 247–259, AMS and International Press, 2001.
- [9] A. Polishchuk, *Rapidly converging series for the Weierstrass zeta-function and for the Kronecker function*, Math. Research Letters 7 (2000), 493–502.
- [10] A. Polishchuk, *Classical Yang-Baxter equation and the A_∞ -constraint*, Advances in Math. 168 (2002), 56–95.
- [11] A. Polishchuk, *A_∞ -algebra of an elliptic curve and Eisenstein series*, Comm. Math. Phys. 301 (2011), 709–722.
- [12] A. Polishchuk, *Moduli of curves as moduli of A_∞ -structures*, arXiv:1312.4636.
- [13] J. Tu, *Homological mirror symmetry and Fourier-Mukai transform*, IMRN 2015, no. 3, 579–630.
- [14] J. Tu, *On the reconstruction problem in mirror symmetry*, Adv. Math. 256 (2014), 449–478.
- [15] A. Weil, *Elliptic functions according to Eisenstein and Kronecker*. Springer-Verlag, 1976.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97403
E-mail address: `apolish@uoregon.edu`